Class XII Miscellaneous (Inverse Trigonometric Functions)

Q.9. prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, $x \in [0,1]$ Date-...

Soln: It is given that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, $x \in [0,1]$

$L.H.S = \tan^{-1}\sqrt{x} = \frac{1}{2}(2\tan^{-1}\sqrt{x})$

$= \frac{1}{2} \times \cos^{-1}\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}$ $\left[\because 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}\right]$

$= \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = R.H.S.$

Q.10. prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$, $x \in \left(0, \frac{\pi}{4}\right)$

Soln: It is given that

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, \; x \in \left(0, \frac{\pi}{4}\right)$$

$$L.H.S = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) \quad \text{—— (I)}$$

Now, $\sqrt{1+\sin x} = \sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}$

$= \sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}$

$\Rightarrow \sqrt{1+\sin x} = \sin\frac{x}{2}+\cos\frac{x}{2}$ $\left[\because \sin^2\frac{x}{2}+\cos^2\frac{x}{2}=1 \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$

Similarly, $\sqrt{1-\sin x} = \cos\frac{x}{2} - \sin\frac{x}{2}$

Then, by (I)

$$L.H.S = \cot^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \left[\frac{\cancel{\sin\frac{x}{2}}+\cos\frac{x}{2}+\cos\frac{x}{2}-\cancel{\sin\frac{x}{2}}}{\sin\frac{x}{2}+\cancel{\cos\frac{x}{2}}+\sin\frac{x}{2}-\cancel{\cos\frac{x}{2}}}\right]$$

$$= \cot^{-1}\left(\frac{\cancel{2}\cos\frac{x}{2}}{\cancel{2}\sin\frac{x}{2}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} = R.H.S.$$

Second Method. miscellaneous. 02-06-2020

Q. 10

$$L.H.S = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$$

$$= \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} \times \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right]$$

$$= \cot^{-1}\left[\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2-(\sqrt{1-\sin x})^2}\right]$$

$$= \cot^{-1}\left[\frac{1+\cancel{\sin x}+1-\cancel{\sin x}+2\sqrt{1-\sin^2 x}}{\cancel{1}+\sin x-\cancel{1}+\sin x}\right]$$

$$= \cot^{-1}\left[\frac{2+2\cos x}{2\sin x}\right] \quad \left[\because \cos^2 x+\sin^2 x=1 \Rightarrow \cos x=\sqrt{1-\sin^2 x}\right]$$

$$= \cot^{-1}\left[\frac{1+\cos x}{\sin x}\right] = \cot^{-1}\left[\frac{\cancel{2}\cos^{\cancel{2}}\frac{x}{2}}{\cancel{2}\sin\frac{x}{2}\cancel{\cos x/2}}\right]$$

$$= \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2} = R.H.S$$

$\left[\because 1+\cos x = 2\cos^2\frac{x}{2} \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$

[You always remember that
if $x \in (0, \frac{\pi}{2})$, then $\sqrt{1-\sin x} = \cos\frac{x}{2} - \sin\frac{x}{2}$
and if $x \in (\frac{\pi}{2}, \pi)$, then $\sqrt{1-\sin x} = \sin\frac{x}{2} - \cos\frac{x}{2}$

Q.11: $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x.$

Prove that

Soln: Let $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$$\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$$

Class - XII Miscellaneous. Date - 02-06-2020

$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2} - \sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2} + \sqrt{2}\sin\frac{\theta}{2}}\right] \quad \left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2},\ 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$

$= \tan^{-1}\left[\frac{\cancel{\sqrt{2}}(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})}{\cancel{\sqrt{2}}(\cos\frac{\theta}{2} + \sin\frac{\theta}{2})}\right] = \tan^{-1}\left[\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}\right]$

$= \tan^{-1}\left[\frac{\cancel{\cos\frac{\theta}{2}}\left(1 - \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)}{\cancel{\cos\frac{\theta}{2}}\left(1 + \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)}\right] = \tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right]$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$

$= \frac{\pi}{4} - \frac{\theta}{2} \quad \left[\because \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}\right]$

$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x. \quad \left[\tan^{-1}(\tan x) = x\right]$

Q. 12. Prove that $\frac{9\pi}{4} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Soln: It is given that $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

$\Rightarrow \frac{9\pi}{8} = \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

$\Rightarrow \frac{9\pi}{8} = \frac{9}{4}\left[\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$

R.H.S $= \frac{9}{4}\left[\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$

$= \frac{9}{4}\left[\sin^{-1}\left\{\frac{1}{3}\sqrt{1-\left(\frac{2\sqrt{2}}{3}\right)^2} + \frac{2\sqrt{2}}{3}\sqrt{1-\left(\frac{1}{3}\right)^2}\right\}\right]$

$\left[\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}\right]$

Scanned with CamScanner

$= \frac{9}{4}\left[\sin^{-1}\left(\frac{1}{3}\times\frac{1}{3}+\frac{2\sqrt{2}}{3}\times\frac{2\sqrt{2}}{3}\right)\right]$

$= \frac{9}{4}\left[\sin^{-1}\left(\frac{1}{9}+\frac{8}{9}\right)\right]$

$= \frac{9}{4}\left[\sin^{-1}(1)\right] = \frac{9}{4}\left[\sin^{-1}\left(\sin\frac{\pi}{2}\right)\right]$

$= \frac{9}{4}\times\frac{\pi}{2} = \frac{9\pi}{8}$

Second method.

L.H.S $= \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\left\{\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)\right\}$

$= \frac{9}{4}\left\{\cos^{-1}\left(\frac{1}{3}\right)\right\}$

$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

$= \frac{9}{4}\sin^{-1}\sqrt{1-\left(\frac{1}{3}\right)^2} \quad \left[\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}\right]$

$= \frac{9}{4}\sin^{-1}\sqrt{1-\frac{1}{9}} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3} =$ R.H.S.

Q.13. Determine, $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

Soln. It is given, $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x) \quad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$

$\Rightarrow \cancel{\tan^{-1}}\left(\frac{2\cos x}{\sin^2 x}\right) = \cancel{\tan^{-1}}\left(2\times\frac{1}{\sin x}\right)$

$\Rightarrow \frac{\cancel{2}\cos x}{\sin^{\cancel{2}} x} = \frac{\cancel{2}}{\cancel{\sin x}} \Rightarrow \frac{\cos x}{\sin x} = 1 \Rightarrow \cot x = 1$

$\Rightarrow \cot x = \cot\frac{\pi}{4}$

$\Rightarrow x = \frac{\pi}{4} \quad \left[\cot\frac{\pi}{4} = 1\right]$

Solved.